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The Building of Mathematical Concepts When Solving Mathematical Tasks in Traditional Way and Using Graphing Calculators

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Annotation. This article aims to understand how students learn mathematical concepts related to the study of functions and differential calculus using graphing calculators with the computer algebra system (GCC) and traditional methods. With a qualitative methodology, we show that students can make the transition to mathematical meanings of concepts using both processes.

Keywords: *symbolic algebraic calculation, differential calculus, reification, semiotic mediation, instrumental genesis.*

Introduction

Since the end of the 20th century, technology has entered people's daily lives, and it is argued that its use in educational contexts should not be neglected. Over the last fifty years, various studies have been carried out on the use of technology in education, both on the use of applications to support teachers' daily work, as a way of simplifying administrative tasks, for example, and to improve students' acquisition of all kinds of knowledge.

It was in the 1980s that, in countries with a higher level of development, teachers and students saw the emergence of graphing calculators with numerical, graphical, and statistical capabilities, and later applications were added that also allowed the study of geometry and the algebraic manipulation of mathematical expressions or Computer Algebra System (CAS). Examples of these tools are the *MAPLE*, *MATHEMATICA*, and *DERIVE* applications, initially aimed at engineers and mathematicians (Kissane, McConney & Ho, 2015).

Research carried out at the end of the last century has shown that both graphic and symbolic representations enable mathematical concepts to be learned effectively (Kendall, 2001; Hannah, 1998).

The aim of this research is to find out how students make the transition to mathematical meanings in the context of problem solving, namely how they incorporate CAS in solving tasks related to functions and differential calculus.

The study reported in this article aims to answer the following research questions:

1. How do students make the transition to mathematical meanings of concepts when solving tasks related to differential calculus, combining traditional methods, and CAS?
2. How is knowledge characterized in the light of some cognitive theories (theory of semiotic mediation (SMT), reification, and instrumental genesis (IG)?

Theoretical Background

This section discusses the use of CAS in secondary school math.

The Semiotic Mediation Theory and Instrumental Genesis are also analyzed, and references are made to Reification Theory.

Computer Algebra System (CAS)

One of the first theoretical studies using CAS was carried out by Kutzler (1994), and the topic was first introduced into maths education at ICME-8 in 1996 in Seville. Following this meeting, several international meetings were organized under the name Computer Algebra in Mathematics Education (CAME), the first of which took place in 1999 in Israel.

For Heid et. al. (2013), the introduction of CAS is intended to help students to develop versatile mathematical thinking, which, according to these authors, involves at least three types of competences:

- The possibility of exchange between representational systems, namely between the perception of a particular mathematical entity as a process and the perception of the entity as an object.
- The exploration of visualization schemes, linking them to logical and analytical schemes.

- Transposition between representations, achieving conceptual and procedural interactions.

These authors refer to the fact that the use of CAS should focus on the interactions between concepts and competences, namely the approach to concepts in which the use of CAS is useful and the thinking and reasoning required by this artefact. CAS allows students to think and reason about the various relationships in mathematics and how they formulate their arguments, and these processes are supported by the objects and tools used.

Lokar and Lokar (2000) distinguish five types of question categories and their respective degree of involvement when asking questions involving CAS, and in practice it can happen that the same task falls into more than one category, depending on the process used to solve it. Thus, the authors consider different types of applications:

T0: *Exercises in which the use of CAS is minimal or non-existent* – in this type of question, the fact that symbolic writing is used leads to a greater expenditure of time than would be necessary if the solution were carried out using traditional methods.

T1: *Traditional exercises* – these are tasks that can be solved either using algebraic manipulation techniques or using symbolic manipulation, which can be solved very quickly using CAS.

T2: *Exercises designed to test the ability to use the CAS* – these are exercises designed to confront the CAS with its own resolution mechanisms.

T3: *Exercises based on a traditional statement* – these are exercises based on a simple problem which, for example, by introducing parameters, can be easily solved with CAS.

T4: *Exercises that are difficult or even impossible to solve using only algebraic manipulation techniques* – in this type of question, the use of algebraic manipulation techniques are somewhat complex, or even unfeasible, and the use of CAS can be an interesting and possible option.

Kendal (2001), in a teaching experiment on differential calculus in the 11th grade using CAS, concluded that both graphical and symbolic representations provide very effective ways of learning mathematical concepts, while Hannah (1998) believes that graphical calculators also make it possible to create rich learning environments for discovering mathematical concepts and that CAS makes students think more deeply in order to discover new mathematical situations.

Currently, CAS is understood more as a means that enables the use of processes related to software systems, which includes symbolic manipulation or the manipulation of expressions with mathematical symbols, and can also be associated with graphical, numerical and/or tabular representations, as well as the use of spreadsheets and/or dynamic geometry programs (Heid et al., 2013; Martins & Domingos, 2020; 2021a; 2021b).

More recently, research has appeared that addresses issues related to assessment at the end of the teaching (Jankvist et al., 2021; Heugl, 2017).

In Portugal, the use of CAS in the classroom is not yet commonplace. However, its use is the subject of advice in the document *Recommendations for Improving Student Learning in Math*, drawn up by the Math Working Group (Carvalho e Silva et al., 2020).

Semiotic Mediation Theory and Instrumental Genesis

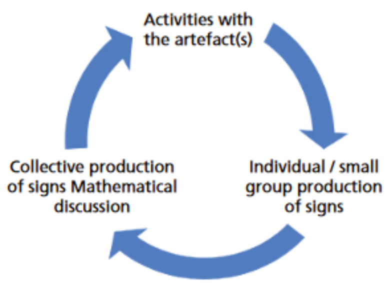
Mediation involves two entities that intersect with the subject, namely mediation with another subject and mediation of the organized learning that is to be established. The absence of any kind of mediation can lead to the acquisition of erroneous knowledge and ineffective procedures (Kozulin et al., 2003).

The integrated whole resulting from the relationships existing with the use of artefacts, for example, GCC, and the performance of the task is called the semiotic potential of the artefact in relation to the task, which is characterized by its facility to associate mathematical meanings evoked by the use of the artefact, which are culturally determined, with personal meanings that each subject develops when using it (instrumented activity) to perform certain tasks.

Analyzing the semiotic potential of a given technological artefact, according to Bussi and Mariotti (2008), involves two points of analysis: one between the artefact and the personal signs that emerge from its use; and the other, between the artefact and the mathematical signs emanating from its use and recognized by an expert as being mathematical, constituting what is known as *Semiotic Mediation Theory* (SMT).

According to SMT, the teacher is responsible for developing the semiotic potential of the artefact. In the collective discussion, the teacher must promote the production of specific and spontaneous signs by the students, which are related to the use of the artefact in conducting the task, guiding the evolution of these signs to obtain the *mathematical signs* (Mariotti & Maffia, 2018). This process, which constitutes semiotic mediation, is based on the iteration of *Teaching Cycles* (figure 1).

Figure 1
The Teaching Cycle



Note. The didactic cycle by Mariotti, M. A., & Maffia, A. (2018, p. 23). From using artefacts to mathematical meanings: the teacher's role in the semiotic mediation process. *Didattica Della Matemática. From Research to Classroom Practice*, *3*, 50–63. https://doi.org/10.33683/ddm.18.4.3.1.

A didactic cycle is nothing more than the realization that there is a relationship between three phases: the activity itself with the artefact or artefacts; the individual or small group production of signs; and the mathematical discussion of the signs.

The various specific characteristics of the artefact and the instrument, plus the processes involved in transforming the artefact into an instrument, are called Instrumental Genesis (IG) (Rabardel, 1995). This creation takes place as the user appropriates the artefact, developing mental schemes involving effective use skills and mobilizing knowledge to make the artefact useful. It is a two-way process: an instrumentalization movement oriented towards the artefact (the artefact is adapted by the subject to their habits and working methods) and an instrumentalization movement centered on the subject (the artefact contributes to structuring the user's activity).

Trouche (2003), reinforcing Rabardel's (1995) arguments, states that a student builds an instrument from an artefact using IG, linking the characteristics of the artefact (potential and restrictions) to their own as a subject (knowledge and work habits). This researcher also emphasizes that:

> This [instrumental] genesis combines processes of instrumentalization and instrumentation. This second process is defined by the appropriation of schemes in order to carry out a certain type of task. The study of the schemes of instrumented action allows us to infer that the student develops their action taking into account their epistemology, their pragmatism and their heuristics. (p. 17)

The construction of an instrument is characterized by a mixed entity, made up of the appropriation of an artefact, material or symbolic, and by the subject, through schemes of use (Rabardel, 1995).

Drijvers and Trouche (2008) consider that this process of appropriation is what makes it possible for the artefact to be "responsible" for mediating the activity. The same authors reinforce the differentiation between two types of use schemas: use schemas, aimed at managing the artefact, such as turning on or adjusting the contrast of a calculator screen, and instrumented action schemas, as actions aimed at carrying out the task, such as calculating the limit of a function. These are coherent and meaningful mental schemas that are constructed from the schemas of use by means of IG.

In short, SMT is what allows the subject to learn when they use an artefact and transform it into an instrument, mobilizing schemas, namely schemas of use and schemas of instrumented action.

Building Mathematical Concepts

For Domingos (2003), interpreting a concept implies looking at that entity with a certain potential, which manifests itself through a sequence of actions.

The theory of *Reification*, according to which mathematical concepts can be conceived in two ways: structurally, as objects, and operationally, as processes (Sfard, 1991), with these two approaches or visions necessarily dependent on each other, For Sfard (1991), and considering that learning does not take place in the same way in all individuals, it is possible to identify three stages in the different learning processes which she called *Internalization, Condensation,* and *Reification*, whose main characteristics are:

1. The first phase, *Internalization* occurs when the subject becomes familiar with the processes that will eventually give rise to a new concept (for example, in the study of functions, it occurs in the manipulation of algebraic expressions, when the student acquires the notion of a variable, whether independent or dependent, by replacing the independent variable in the analytical expression of the function with one or more values).
2. The second phase, *Condensation* corresponds to the moment when the subject understands the sequences of processes, developing the ability to think about a given process (for example, in the study of functions, it corresponds to the stage when the student finds it easy to switch between different representations). For Sfard, this moment is crucial in the construction of the concept.
3. The third and final phase, *Reification,* occurs when the student can 'see' a new mathematical entity as a complete and autonomous object, in the form of an integrated whole, already removed from the processes that gave rise to it.

Once a concept has been reified, it can serve as the basis for the formation of new concepts at a higher level. By becoming aware of the existence of a new mathematical object, the student can start a new learning cycle, restarting another process through the *Internalization* phase, which will culminate in the *Reification* of this new entity (Sfard, 1991) .

Ruthven et al. (1997) say that CAS plays a positive role in reorganizing thinking as a cognitive tool, enabling the development of a given student's working capacities when faced with non-routine problems, as well as providing interactive learning environments and a greater possibility of broadening ways of thinking.

CAS, more than an artefact, should be seen as a platform on which the constructivist principles are based, especially when the aim is for students to learn concepts related to algebra and analysis.

Methodology

The approach used is qualitative, following the interpretive paradigm and based on a case study.

According to Bogdan and Biklen (1994), the data collected is qualitative when it is rich in descriptive details that include people, places, and conversations, and its statistical treatment is complex and difficult to carry out. The questions being investigated do not depend on the operationalization of variables, and the main aim of this type of research is to analyze the whole process in its fullness and complexity, given the natural context in which the phenomena occur. It involves the systematic observation of processes, both informal, such as interviews, and formal, such as questionnaires and documentary data.

For Yin (2012), a case study is an investigation based on fieldwork, studying a person, program or institution in its context, using semi-structured interviews, observation, documents, questionnaires, and artefacts. Case studies are often used when events cannot be controlled, and it is therefore not possible to manipulate the causes of the participants' behavior.

The presentation of the results analyzes the joint actions of two students, with the fictitious names of Antónia and André, in solving a task in pairs. These two students, both 17, who had known each other since the 8th grade, had different academic backgrounds, although they were both motivated to take part in the study.

They were in the 12th year of school, in the humanistic science and technology course, in a class of 27 students, in a school in the greater Lisbon area, in Portugal.

They had good oral and written communication skills and good behavior. Antónia is a student with an average academic performance, and André is a successful student in Mathematics, with grades in the excellent range.

The descriptive data collection methods were based on the students' written reports, direct observation, images of the GCC screen representations, and the logbook (Bogdan & Biklen, 1994) and aimed to identify the students' reactions during their interaction with the calculator for the CAS application to learn math using the constructivist theory of *Reification* (Sfard, 1991) and SMT as tools for analysis.

In this study, the production of *artefactual signs* is encouraged through the development of *schemas of use* and *schemas of instrumented action,* in the communication developed between the researcher and the students and in the oral and written communication of the students together. During the process, a mathematical discussion was encouraged between the two students, where the researcher promoted the transition from *artefactual signs* to *mathematical signs,* supported by the schemas. Finally, a summary of the analysis of the results is presented, with the aim of answering the research questions initially formulated.

The Tasks

One of the tasks presented to students Antónia and André (Figure 2), which we call task 1, comes from the Mathematics Curriculum Programme and Guidelines for Secondary Education (Bivar et al., 2014). Task 1 was devised by the researcher, and some of the content covered refers to content from the 10th grade the on topic of Real Functions of Real Variable, namely solving problems involving the geometric properties of Real Functions of Real Variable graphs, and the rest to the 12th year, Derivatives of Real Functions of Real Variable and Applications and Bolzano-Cauchy Theorem.

Figure 2

Task 1 Presented to Students Antónia and André

Consider the function f of domain $\mathbb{R}$ defined by:

$$f(x) = x^3 + ax^2 - bx \text{ where } a, b \in \mathbb{R}$$

It is known that the graph of f has an inflection point with coordinates $(1, -1)$

a) Show that $a = -3$ e $b = -1$

b) Prove that f has at least one zero in $]2, 3[$ and then determine it by giving a value rounded to the nearest hundredth .

c) Determine the monotony intervals and relative extremes of f

d) The equation $f(x) = x^2$ has exactly two solutions in $\mathbb{R}^+$. Find the values to the nearest tenth of these solutions.

The aim of this task was to see how the students would solve the proposed exercises, whether they would use only algebraic manipulation or only symbolic manipulation, i.e., using the GCC, based on the theoretical assumptions mentioned above.

A second task, which Antónia and André carried out and whose resolution is analyzed in this article, is shown in Figure 3, which we call Task 2.

Figure 3

Task 2 Presented to Students Antónia and André

It is said that Ferrari and Tartaglia, 16th century Italian mathematicians, took part in a famous public debate where, among many other challenges, Ferrari posed the following problem to Tartaglia:

> "How to divide 8 into two parts so that the product of these parts is multiplied by the difference between them is as large as possible?"

a) Solve the problem analytically and prove that the exact results are:

$$8\left(\frac{3-\sqrt{3}}{6}\right) \text{ e } 8\left(\frac{3+\sqrt{3}}{6}\right)$$

b) What if the number was 9, 10, ..., n? What parts would you divide the number into?

(Adapted from Graphic Technology in the future Mathematics A program in secondary education, T3 Portugal)

The aim of this second task was for students to create a mathematical model and work on it, first using analytical processes, analyzing a simple case, and then using GCC, making possible generalizations.

Presentation and Analysis of Data

In this section, the data collected is presented, and a descriptive analysis of this data is conducted.

Artefacts Used

The two tasks analyzed here were solved by students Antónia and André in the 2nd term of the 2018/2019 school year. In solving the tasks, both students developed *schemes of instrumented action* based on *mathematical signs* they knew related to the concepts involved.

The calculators used by the students were TI-nspire *CX* CAS™, attached to *cradles* so that they could communicate wirelessly with the TI-Navigator application (Figure 4) so that the researcher could assess the work done by the students when they used the calculator.

Figure 4
TI-nspire CX CAS Calculators Connected to the TI-Navigator Application

Task 1

Relatively to the first question of Task 1, which sought to show that the parameters of the function under study would be $a = -3$ e $b = -1$ the aim was to discuss concepts involving the image of a function at a point and the inflection point of the graph of a function, determined using the second derivative of a function. After the students in this group had thought about the question for a while, they engaged in the following dialogue:

Antónia:	That is the first one.
Researcher:	So how did you do it?...
Antónia:	Replacing and for the coordinates. The x for 1 and y by -2.
Researcher:	Right.
Antónia:	Ah! Then we concluded that $a = -2 + b$. And we substituted the values we were given to check. Can't we do it like this?
Researcher:	Is that so ? (...) So you have looked at the image. Where is the inflection point? How do you know that is an inflection point?
André:	Because it must be a zero.

Researcher: Zero?

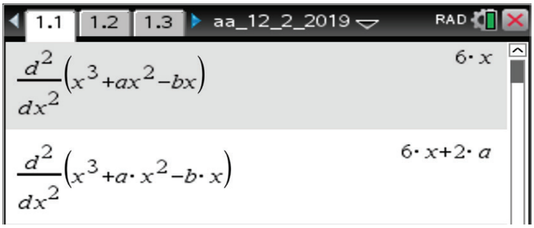
André: From the second derivative... The second derivative at 1 must be zero.

As can be seen from the dialogue, there was a need to establish a return-to-task action that preceded a focusing and redirecting action.

The students solved this question using two different processes: a first process in which they used GCC to obtain the expression for the second derivative of the family of functions *f* (Figure 5) and simultaneously using traditional methods to obtain the parameters *a* e *b* with a system; and a second process, using only GCC.

Figure 5

Obtaining the Second Derivative



In terms of IG, and as we can see in the first line of the calculator screen in Figure 5, when the students wrote the symbolic form of the second derivative of the function *f*, they made an inaccuracy in the introduction of the instruction required which resulted in an expression in which the constant does not 'appear' *a* as would be expected, which is perfectly common in this model of calculator because the multiplication operation between *a* e x^2 and between *b* e *x*. This discrepancy, which can be verified very easily by mental calculation, was analyzed and rectified by the students, as can be seen in the following dialogue:

André: The 2nd derivative is not .

Antonia: You are right...

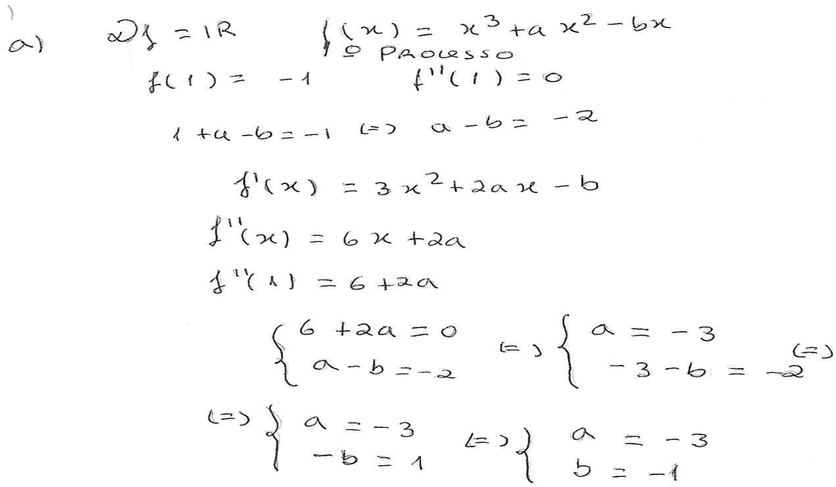
Researcher: Look carefully at the expression introduced.

André: Is multiplication missing?

By failing to introduce instruction of the second derivative, students are somehow forced to interpret not only the instrument they are working with, but also the meaning of the second derivative in the GCC and thus be able to give mathematical meaning to the instruction as Bussi and Mariotti (2008) say. In parallel, according to Sfard (1991), students are facing a process of *internalization*, followed by a process of *condensation*. Rectified the problem, the students then overcame the issue using algebraic manipulation, as shown in Figures 6.

Figure 6

First Resolution of Point a) of Task 1 Presented by Students Antónia and André


a) $D_f = \mathbb{R}$ $f(x) = x^3 + ax^2 - bx$

1º PROCESSO

$f(1) = -1$ $f''(1) = 0$

$1 + a - b = -1 \Leftrightarrow a - b = -2$

$f'(x) = 3x^2 + 2ax - b$

$f''(x) = 6x + 2a$

$f''(1) = 6 + 2a$

$\begin{cases} 6 + 2a = 0 \\ a - b = -2 \end{cases} \Leftrightarrow \begin{cases} a = -3 \\ -3 - b = -2 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} a = -3 \\ -b = 1 \end{cases} \Leftrightarrow \begin{cases} a = -3 \\ b = -1 \end{cases}$

The students began by performing the first and second derivatives of the family of functions f using the GCC. This artefact essentially served to support the algebraic resolution presented, to confirm what the students apparently already knew using their knowledge of the derivatives of polynomial expressions and mental calculation, as can be seen from the students' discourse. They recognized that the coordinate point (1, -1) is the inflection point of the graph of the family of functions as shown in Figure 6, correctly using the fact that they know that $f(1) = -1$ and $f''(1) = 0$.

Based on the analysis of Figure 6, and considering the resolution presented, it can be deduced that the students have these constructs reified (Sfard, 1991), i.e., they know that the analysis of the zeros of the second derivative is used to obtain the inflection points. having this knowledge has repercussions in the handling of the GCC in the form of instrumented action (Drijvers & Trouche, 2008).

Figure 7 shows the instruction given in the GCC and its result to obtain the zero of the second derivative of f in which the students work with parameters, also showing that they know how to generalize concepts related to the notion of derivative and its implications. This second resolution is symbolic, and although it wasn't requested by the researcher, the students felt the need to carry it out as a way of proving their initial resolution.

Figure 7

Obtaining the Zero of the Second Derivative of f

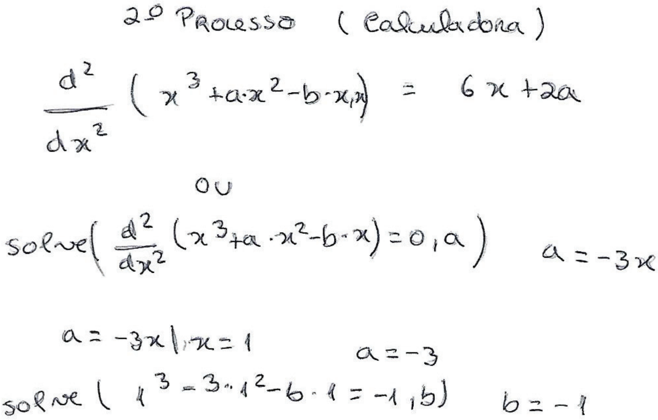
Based on the expression obtained, the students transcribed the instruction onto paper (Figure 8), and continuing to use the GCC, they obtained the same result as before. However, this time with a more persistent use of the GCC.

It should be noted that the notation used by the students is different from the one they usually use (two apostrophes), revealing perfect condensation (Sfard, 1991) of the representation used in the artefact ($\frac{d^2}{dx^2}(x^3 + a.x^2 - b.x, x)$).

On the other hand, the students are conditioned by the GCC. Their mental images are based on what the GCC provides on the screen, both in terms of the representation presented by the second derivative and even when they present the zero of the second derivative in order of the variable *a* instead of the variable *x* as was usual.

Figure 8

Written Resolution of Point a) of Task 1, Using Only Symbolic Manipulation



Using the semiotic potential of the GCC, these two students were able to mobilize concepts related to the 2nd derivative in solving this question, confirming the values of the constants *a* e *b* expressed in the statement. They initially present an analytical solution, based on traditional methods and memorization, using the GCC apparently to confirm known results, and a second resolution in which they resort solely and exclusively to the GCC.

The construction of the answer given by the students and the processes involved in acquiring knowledge about the second derivative of a function allow us to recognize the three phases of reification theory – internalization, condensation, and reification (Sfard, 1991), and a perfect IG by the two students. As Bussi and Mariotti (2008) say, it can be seen by comparing the different products presented by the students (Figures 7 and 8), that they are learning to use the GCC and interact with the artefact, representing the instruction of the zeros of the second derivative differently in each of the figures, showing a good condensation (Sfard, 1991) of the concept. The GCC forces the students to think differently so that they can take advantage of the CAS.

In fact, there is a difference in approach between what the students present in Figure 6 and what they present in Figure 8. The first figure shows a purely analytical approach, based on what they had learnt in class. In the second figure, the approach used by the students is one of adaptation to the GCC in which the students internalize and establish connections between the representations they have previously learned and the representations provided by the GCC.

This task is of type T1 (Lokar & Lokar, 2000), since the use of CAS, as can be seen, could perfectly well have been replaced by traditional methods.

When it comes to demonstrating that f has at least one zero in $]2, 3[$ the students began by using the GCC to determine the images of 2 and 3 using f (Figure 9).

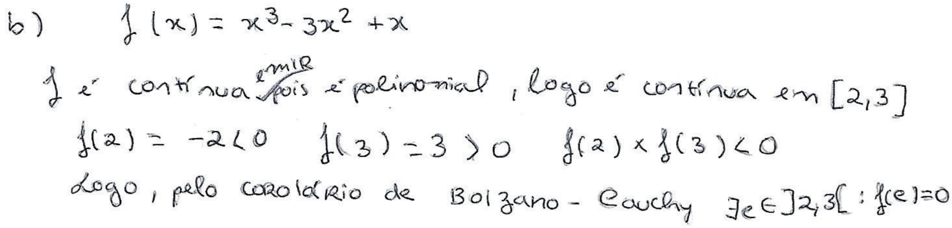
Figure 9

Images of 2 and 3 by f

Afterwards, they applied the corollary of the Bolzano-Cauchy theorem (Figure 10), as planned, a concept that had been taught during the lessons.

Figure 10

Applying the Bolzano-Cauchy Theorem to Solve Point b) of Task 1



b) $f(x) = x^3 - 3x^2 + x$

f é contínua em IR pois é polinomial, logo é contínua em $[2,3]$

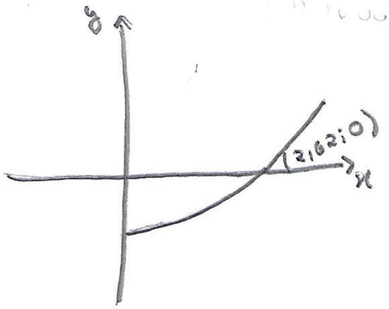
$f(2) = -2 < 0$ $f(3) = 3 > 0$ $f(2) \times f(3) < 0$

Logo, pelo corolário de Bolzano - Cauchy $\exists c \in]2,3[: f(c)=0$

In order to determine the zero in the indicated interval, giving a value rounded to the nearest hundredth, the students used the GCC to sketch the graph of f by transposing the graph obtained onto paper (Figure 11), considering the values of a and b indicated in the task.

Figure 11

Graph from for e



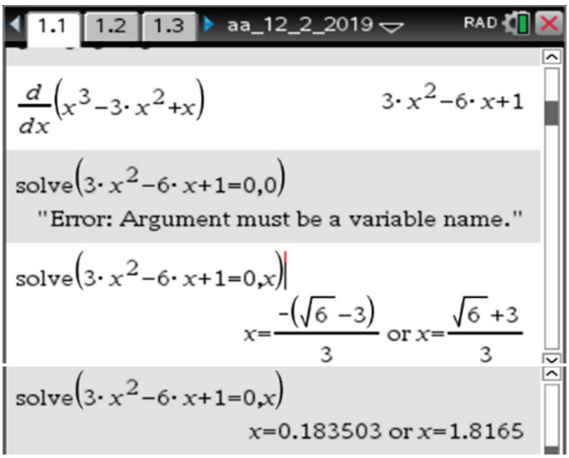
The students only used the calculator's graphical capabilities and did not use any kind of symbolic manipulation that explicitly involved using CAS on the graphing calculator, which, according to Heid et al. (2013), shows that the students are using versatile mathematical thinking, since they could easily have constructed an instruction to determine the zero in question.

On the other hand, the students didn't come to any kind of conclusion, although the solution is presented as the value of the abscissa of the coordinates of the point that results from the intersection of the graph of the function with the abscissa axis. This task is of the T0 type (Lokar & Lokar, 2000), since there is no explicit use of CAS.

Regarding the question in which they wanted to determine the monotonic intervals of f and to study the existence of relative extremes, the students only used GCC, calculating the expression for the derivative of f and determining its zeros. First in an exact way and then in an approximate way (Figure 12), with Antónia being responsible for entering the symbols into the calculator.

Figure 12

Obtaining the Derivative of And Its Zeros Using a Calculator



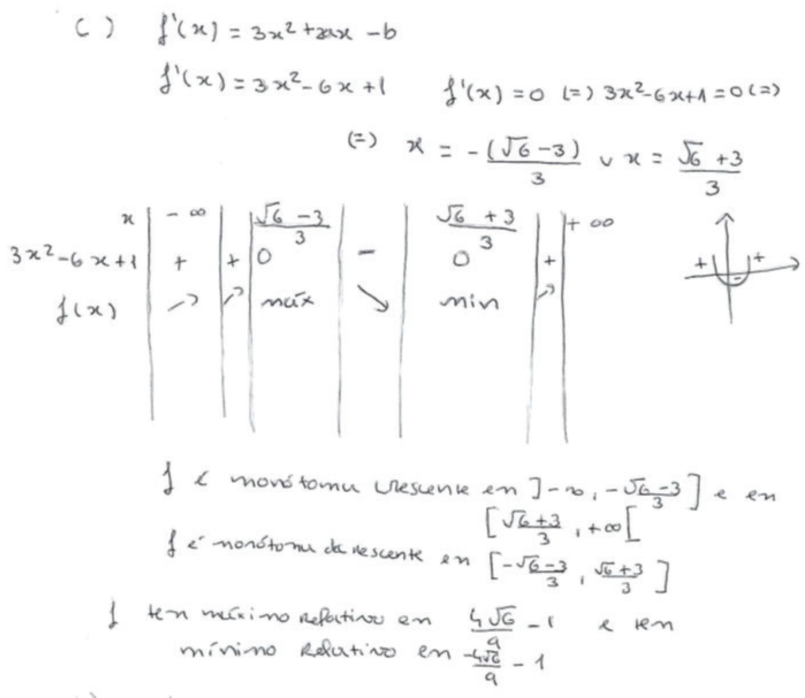
In terms of IG, the derivative is correctly entered into the calculator. However, when the students write the instruction that allows the zeros to be calculated, it was not entered correctly, a situation that was corrected by the student, without any intervention from the researcher, thus demonstrating a good understanding of the representation in the calculator for calculating the zeros of the associated expression $(3x^2 - 6x + 1)$.

The fact that the students had obtained the values in two forms: exact and approximate, gave them an idea of the location of the derivative's zeros on the real line, thus implementing appropriate instrumented action schemes (Drijvers & Trouche, 2008), namely using GCC's Algebra menu.

Afterwards, André finalized the resolution by complementing it with the sign chart and the monotony intervals of f (Figure 13).

In addition to showing the derivative of the function f and the zeros of f' in the representation of the sign table of f' and variation of f, it shows two columns for values less than $-\frac{\sqrt{6}+3}{3}$, as if they understood $-\infty$ as a finite number. This is not the case for values greater than $\frac{\sqrt{6}+3}{3}$ which lead us to believe that it was simply an oversight in the representation of the sign board.

Figure 13
Conclusion of Point c) Presented in the Task



André still feels the need to sketch a diagram of the parabola to complete the sign chart when he could just as easily use the GCC to obtain the value of the derivative at a given point and thus conclude on the sign of the derivative in the interval where the chosen point is. In this way, he does not produce a solution in which he only uses CAS, but instead needs an analytical crutch, i.e., traditional methods.

On the other hand, André does not think it is enough for the expression of the 1st derivative to have two zeros and the coefficient of the term with the highest positive degree to conclude the monotony intervals of *f* and the relative extremes of the function.

The resolution method presented by the students is type T1, in which the calculator is used instead of the derivation rules applied to a polynomial function. The two processes used do not rely exclusively on the students' knowledge of the derivation rules. The GCC serves here as a complement to an algebraic resolution, in which the students are aware of the steps they must take algebraically to solve the question posed and carry out the question in a second approach using the GCC exclusively, as if they needed confirmation of the process they regularly use.

Looking at Figure 13 in more detail, we can see that these students have consolidated the mathematical signs of the study of the monotony of a function, and have fully mastered both the analytical process, done with traditional methods, and the process in which they use GCC exclusively. In Figure 13, the students summarize what they have obtained both analytically and using CAS, as there is no real concern on their part to present the transition steps between the various stages as would usually be required in an integral analytical solution.

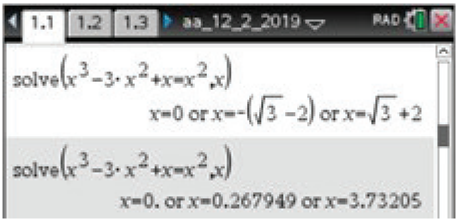
The students appear to have a full understanding of the process to be used, even though they have obtained the derivative and zeros using the GCC and constructed a table describing the situation. The reification stage (Sfard, 1991) has been reached, as the students present a completely integrated study of the function.

It is also possible to see that they manage to generalize the concept inherent in the study of the monotony of a function since they carry out the expected study for the particular case of the function by basing the sign of the derivative on the graph of the derivative represented next to the table. It's as if the sketch of the graph worked as a mnemonic to obtain the sign of the derivative.

The next point was to obtain approximate solutions to the equation $f(x) = x^2$ in $\mathbb{R}^+$. a question of type T1. Antónia and André solved it using symbolic manipulation, developing instrumented action schemes (Figure 14), namely as a way of obtaining the solutions of the equation $x^3 - 3x^2 + x = x^2$. First, they used a GCC option that allows them to obtain exact values and then, by selecting the option to obtain approximate values, they achieved what they wanted. In both cases, they used the Algebra option in MENU, showing a perfect IG.

Figure 14

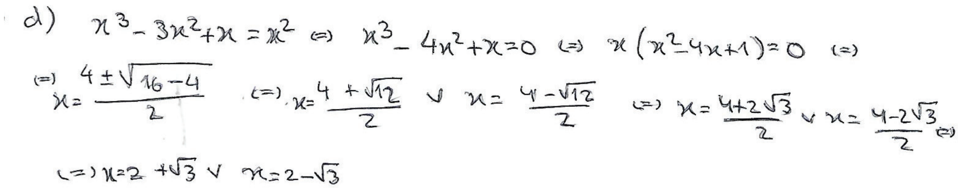
Solving Point d) Symbolically With Exact and Approximate Values



In this case, and again as a way of confirming the solution presented, as if there was an intrinsic need to feel safe, the students felt the need to carry out a second purely analytical process (Figure 15), as if the fact that the GCC gave them the values of the equation wasn't enough at all.

Figure 15

Solve Point d) Analytically



The resolution presented using traditional methods, although correct, as it shows the two possible solutions in $\mathbb{R}^+$ (Figure 15), could have led the students to doubt the previous result, as they should have entered the following instruction in the GCC:

$$solve(x^3 - 3.x^2 + x = x^2, x)|x > 0$$

By not doing so, the students neglected the solution $x = 0$ possible in $\mathbb{R}$ but impossible in the set considered.

The way these two students solved this task allows us to conclude that they know how to use the mathematical signs needed to solve the task in an appropriate way and that they manage to make a good transposition between algebraic and symbolic representations, even in situations where they are not required to present different processes, using schemas of use and schemas of instrumented action when working with the calculator and CAS, exploring visualization schemas, and linking them to logical/deductive schemas (Drijvers & Trouche, 2008).

Task 2

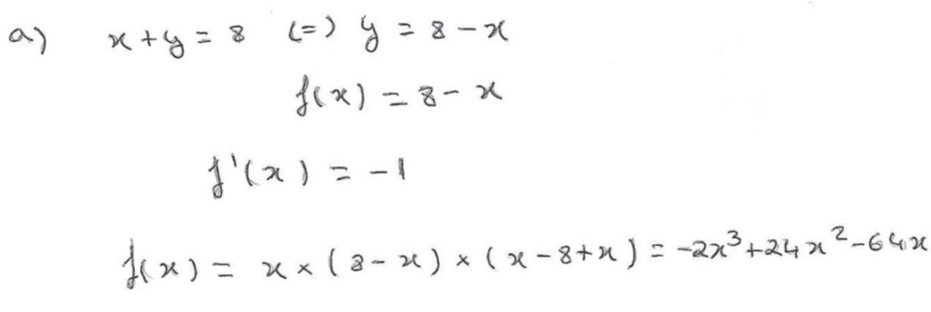
In this task, and in the first question, the aim is to divide the number 8 into two parts so that the product of these parts multiplied by the difference between them is as large as possible. In the first part, students should develop a mathematical model, and in the second part they should generalize the specific situation, using GCC.

Regarding the first question, where you want to solve the problem analytically and prove that the exact results $8\left(\frac{3-\sqrt{3}}{6}\right)$are $8\left(\frac{3+\sqrt{3}}{6}\right)$ and there are two phases to solving the problem. The first phase consists of building the model that allows the situation to be solved and the second consists of obtaining the values that answer the specific question.

Antonia:	What's this about?
André:	[After a few minutes] We have to multiply one by the other and multiply again by subtracting the two.
Researcher:	How much do the parts have to add up to? If one is how much is the other?
Antónia:	It is $8 - x$.
Researcher:	What is your expression? What do you want to get? Read the statement... What is the product?
Antónia:	$x \times (8 - x)$
Researcher:	And this product has to be multiplied by?...
André:	$x \times (8 - x) \times [x - (8 - x)]$

After the dialogue, the students proceeded to build a model on the answer sheet (Figure 16).

Figure 16
Construction of the Model in Analytical Form



As we can see by analyzing Figure 16, the students began by writing the model describing the situation without feeling the need to use the GCC.

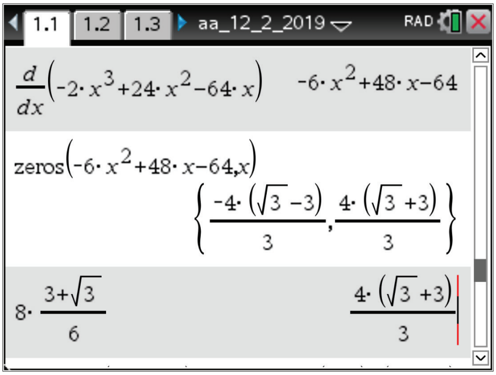
Regarding the second part of the initial question,

Researcher: What is the biggest possible thing?
André: It is great.
Researcher: And how do you get it?
Antónia: We derive and calculate the zeros...

After this dialogue, the students determined the expression of the derivative and its zeros using GCC and verified that the values in the statement were true, as we can see (Figure 17).

Figure 17

Determining the Derivative, its Zeros and Confirming the Result Given in the Statement



The students developed *instrumented action schemes* (Drijvers and Trouche, 2008), typing into the calculator the instructions needed to obtain the derivative expression and its zeros, thus arriving at the values presented to them in the task statement demonstrating a perfect IG of the process required to solve the problem.

This task can be classified on the Lokar and Lokar (2000) scale as type T2, since it is the result of a traditional statement which tests the ability to use CAS by exploring more abstract expressions, indicating a generalization of the initial model.

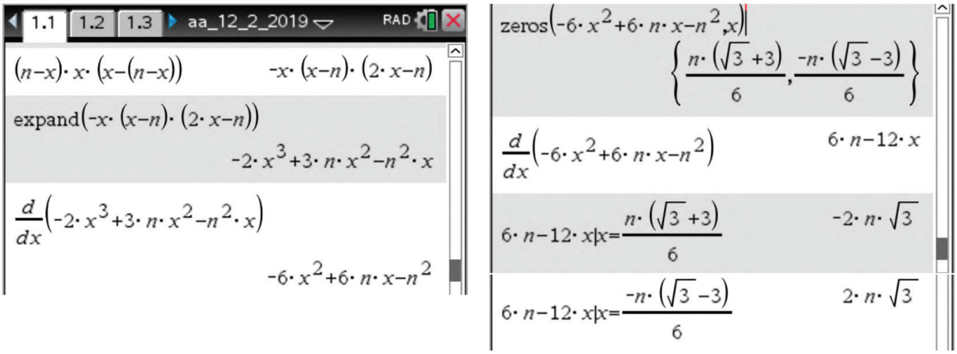
The GCC allows students to reach conclusions more quickly using procedures identical to those they would have used if they had solved the problem using traditional methods, namely determining the derivative and its zeros and checking that the values obtained are identical to those indicated in the statement, using the GCC.

On the other hand, as in the previous task, the students manage to accommodate the different representations of the derivative, both the way it was initially learned in class and the representation provided by GCC, showing that the students have reached the three phases of the reification theory (Sfard, 1991).

Relatively to the second part of the question, in which students were asked to analyze which parts the number would be divided into if it was 9, 10, ..., n, the students could have tested other values in addition to the value tested in the previous point, however, they chose to introduce a parameter (n) into their solution. In this case, the students adopted a similar strategy to the one adopted in the previous case, substituting 8 for *n*, as can be seen in the calculator screen shown in Figure 18, in which they developed the procedures leading to the generalization of the process.

Figure 18

Symbolic Resolution of Point b) of Task 2



They have constructed an expression that results from the product of the two parts, x and $n - x$ multiplied by the difference between x and $n - x$ and then expanded the resulting expression using the *expand()* command, probably to familiarize themselves with the polynomial form they are normally used to working with, particularly when using purely analytical processes.

They then obtained the derivative of the expression thus obtained, using GCC, and calculated its zeros. Finally, they arrived at an expression for the second derivative and depending on whether the sign was positive or negative, they concluded that the image of the zeros of the derivative of the expression obtained is minimum or maximum, respectively.

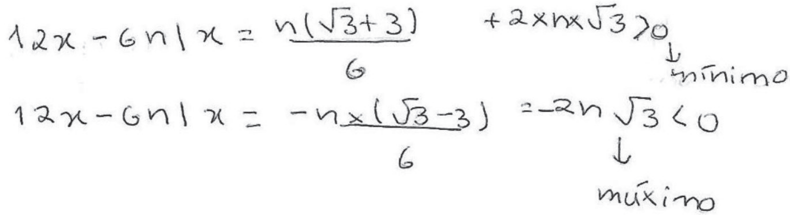
In this case, the students were able to generalize and solve the problem using only symbolic manipulation in the GCC. As can be seen in Figure 18, the students went on to analyze whether the values they obtained for x were maximizing or minimizing, applying previously acquired knowledge by applying the second derivative test, as we can see from André's comment.

André: If the second derivative at the point is positive it will be a minimum and if it is negative, it will be a maximum.

Transposing the processes conducted in the GCC onto paper (Figure 19) in the form of a conclusion helps to favor mathematical communication and thus allows a better understanding of the situation to be solved. The students' understanding of the different representations involved, both in the GCC and in the exclusively analytical process, allows them to improve their notion of what it means to do math's.

Figure 19

Conclusions Presented by students Antónia and André on Point b) of Task 2



When presenting their conclusions, and as you can see from looking at Figure 19, the students present the extremes, claiming that they are extremes, while the extremes are the images of these values. This is a slip of the tongue, as the students have already shown that they know the difference between extreme and extremum.

The resolution of the tasks presented allows us to assume that these two students, interacting with each other, used the *mathematical signs* considered appropriately and managed to transpose them well between the representations available to them, namely by correctly making the transition from analytical processes to processes that include symbolic manipulation in the GCC, and vice versa, using *schemas of use and* schemas of *instrumented action* (Drijvers & Trouche, 2008) when working with the artefacts, calculator and CAS, exploring visualization schemas, and linking them to logical and analytical schemas. SMT is seen here, as the students use the Calculator, and CAS artefacts to produce mathematical signs as Bussi and Mariotti (2008) say.

The appropriate use of the symbology associated with the calculator, and in particular CAS, is a factor that encourages the acquisition of appropriate writing of mathematical symbols, thus enabling the *reification* of concepts (Sfard, 1991).

Conclusions

The aim of this research is to find out how students interact with GCC and how they incorporate CAS into the study of tasks related to functions and differential calculus.

As we noted above, this study aims to answer the following research questions:

1) How do students make the transition to mathematical meanings of concepts when solving tasks related to differential calculus, combining traditional methods and CAS?
2) How is knowledge characterized in the light of some cognitive theories (theory of semiotic mediation (TMS), Reification, and instrumental genesis)?

As this study is based on a case study, its reliability is obtained if other researchers arrive at identical results using the same methodologies carrying out the same type of research.

Relatively to the 'generalization' of the conclusions and results of a case study, it is essential to understand that the research methodology used is not intended to generalize the results obtained, but rather to provide in-depth knowledge of specific cases, as Merriam (1988) and Yin (2012) say.

The use of GCC allows the two points of analysis highlighted by Bussi and Mariotti (2008) in their SMT to be verified, since not only do the students use the technological artifact, interpreting its signs and transforming it into an instrument, but they also use it to learn topics related to functions and differential calculus.

The use of CAS in this teaching experience suggests that students, when they get to grips with GCC, can take different approaches to solving the same problem, whether they are symbolic, analytical, and symbolic, or graphical, even in situations where this was not requested or apparently not necessary.

The students have a perfect grasp of the different representations of the same concept, whether expressed on the GCC or with traditional methods, and can use the GCC to complement and reinforce the ideas and processes developed analytically. As Drijvers and Trouche (2008) say, in the work carried out by the two students, schemes of instrumented action were mobilized, namely when the students used various mathematical functions, such as the 'Solve' command to calculate the solutions to an equation, or the 1st or 2nd derivative commands; and schemes of use, when they knew, for example, which keys they had to press to be able to change the display window of function graphs or make a simple copy of an expression. They showed that they didn't have to repeat it endlessly, showing an excellent command of the use of differential calculus in this type of problems.

These two students have also consolidated the mathematical signs relating to the Bolzano-Cauchy theorem and its demonstration of the existence of a solution to an equation on a given interval passing from the phases of internalization to reification (Sfard, 1991). This occurs also in the study of the monotony of a function and the study of the direction of the concavities of the graph of a function, managing to establish relationships between different types of representations.

It also can be seen that the discourse between the two students promoted the production of specific signs related to the graphing calculator and CAS artefacts, leading to the acquisition of mathematical signs related to functions and differential calculus, and that the stage of reification was apparently reached (Sfard, 1991), after taken previously the two phases of *internalization* and *condensation,* which can be seen in the fact that the students studied the proposed function in an integrated way and in the resolution presented by the students in task 2. In this sense, it was beneficial that the two tasks were conducted in pairs, given the interactions taken between the two students.

In short, the analysis of the resolutions shown by these two students, working with GCC and traditional methods side by side, reinforces the learning of mathematical concepts related to the study of functions and differential calculus, allowing also the students to accommodate and transpose between different types of representation, some of which are unfamiliar in their day-to-day work with the subjects of study.

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Matematinių sąvokų kūrimas sprendžiant matematinius uždavinius tradiciniu būdu ir naudojant grafinius skaičiuotuvus

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Santrauka

Atsižvelgiant į grafinio skaičiuotuvo artefaktą, integruotą su kompiuterinės algebros sistema (GCC), šio straipsnio tikslas išanalizuoti, kaip mokiniai įsisavina matematines sąvokas spręsdami matematinius uždavinius problemų sprendimo kontekste. Straipsnyje analizuojamas dviejų dvyliktos klasės mokinių dviejų uždavinių, susijusių su funkcijomis ir diferencialiniu skaičiavimu, sprendimas naudojant GCC ir tradicinius metodus, pagrįstus reifikacijos, instrumentinės genezės ir semiotinės mediacijos teorijomis.

Tai kokybinis interpretacinės paradigmos tyrimas, paremtas atvejo tyrimo metodologija, integruotas į platesnį tyrimą, kurio tikslas – suprasti, kaip mokiniai savo praktinėje veikloje naudoja tradicinius metodus ir grafinius skaičiuotuvus su integruota kompiuterine algebros sistema.

Duomenų analizė rodo, kad mokiniai sugeba įsisavinti matematines sąvokas, susijusias su funkcijų ir diferencialiniu skaičiavimu, naudodami simbolines manipuliacijas ir susiedami jas su tradiciniais metodais, taip demonstruodami matematinių sąvokų reifikaciją.

Esminiai žodžiai: *simbolinis algebrinis skaičiavimas, diferencialinis skaičiavimas, reifikacija, semiotinė mediacija, instrumentinė genezė*.

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